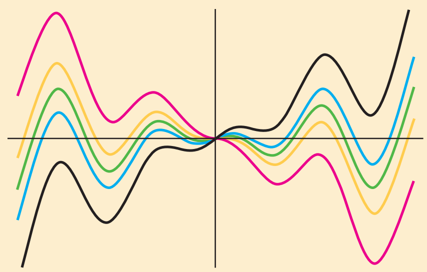
# 1 INTRODUCTION TO DIFFERENTIAL EQUATIONS





The words *differential* and *equations* certainly suggest solving some kind of equation that contains derivatives $y', y'', \ldots$. Analogous to a course in algebra and trigonometry, in which a good amount of time is spent solving equations such as $x^2 + 5x + 4 = 0$ for the unknown number $x$, in this course *one* of our tasks will be to solve differential equations such as $y'' + 2y' + y = 0$ for an unknown function $y = \phi(x)$.

The preceding paragraph tells something, but not the complete story, about the course you are about to begin. As the course unfolds, you will see that there is more to the study of differential equations than just mastering methods that someone has devised to solve them.

But first things first. In order to read, study, and be conversant in a specialized subject, you have to learn the terminology of that discipline. This is the thrust of the first two sections of this chapter. In the last section we briefly examine the link between differential equations and the real world. Practical questions such as *How fast does a disease spread? How fast does a population change?* involve rates of change or derivatives. As so the mathematical description—or mathematical model—of experiments, observations, or theories may be a differential equation.

# 1.1 DEFINITIONS AND TERMINOLOGY

## REVIEW MATERIAL

- Definition of the derivative
- Rules of differentiation
- Derivative as a rate of change
- First derivative and increasing/decreasing
- Second derivative and concavity

**INTRODUCTION** The derivative $dy/dx$ of a function $y = \phi(x)$ is itself another function $\phi'(x)$ found by an appropriate rule. The function $y = e^{0.1x^2}$ is differentiable on the interval $(-\infty, \infty)$, and by the Chain Rule its derivative is $dy/dx = 0.2xe^{0.1x^2}$. If we replace $e^{0.1x^2}$ on the right-hand side of the last equation by the symbol $y$, the derivative becomes

$$\frac{dy}{dx} = 0.2xy. \tag{1}$$

Now imagine that a friend of yours simply hands you equation (1)—you have no idea how it was constructed—and asks, *What is the function represented by the symbol y*? You are now face to face with one of the basic problems in this course:

*How do you solve such an equation for the unknown function $y = \phi(x)$?*

**A DEFINITION** The equation that we made up in (1) is called a **differential equation.** Before proceeding any further, let us consider a more precise definition of this concept.

**DEFINITION 1.1.1 Differential Equation**

An equation containing the derivatives of one or more dependent variables, with respect to one or more independent variables, is said to be a **differential equation (DE).**

To talk about them, we shall classify differential equations by **type, order,** and **linearity.**

**CLASSIFICATION BY TYPE** If an equation contains only ordinary derivatives of one or more dependent variables with respect to a single independent variable it is said to be an **ordinary differential equation (ODE).** For example,

A DE can contain more than one dependent variable

$$\frac{dy}{dx} + 5y = e^x, \qquad \frac{d^2y}{dx^2} - \frac{dy}{dx} + 6y = 0, \qquad \text{and} \qquad \frac{dx}{dt} + \frac{dy}{dt} = 2x + y \tag{2}$$

are ordinary differential equations. An equation involving partial derivatives of one or more dependent variables of two or more independent variables is called a

**partial differential equation (PDE).** For example,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} - 2\frac{\partial u}{\partial t}, \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{3}$$

are partial differential equations.*

Throughout this text ordinary derivatives will be written by using either the **Leibniz notation** $dy/dx, d^2y/dx^2, d^3y/dx^3, \ldots$ or the **prime notation** $y', y'', y''', \ldots$. By using the latter notation, the first two differential equations in (2) can be written a little more compactly as $y' + 5y = e^x$ and $y'' - y' + 6y = 0$. Actually, the prime notation is used to denote only the first three derivatives; the fourth derivative is written $y^{(4)}$ instead of $y''''$. In general, the $n$th derivative of $y$ is written $d^ny/dx^n$ or $y^{(n)}$. Although less convenient to write and to typeset, the Leibniz notation has an advantage over the prime notation in that it clearly displays both the dependent and independent variables. For example, in the equation

unknown function
or dependent variable

$$\frac{d^2x}{dt^2} + 16x = 0$$

independent variable

it is immediately seen that the symbol $x$ now represents a dependent variable, whereas the independent variable is $t$. You should also be aware that in physical sciences and engineering, Newton's **dot notation** (derogatively referred to by some as the "flyspeck" notation) is sometimes used to denote derivatives with respect to time $t$. Thus the differential equation $d^2s/dt^2 = -32$ becomes $\ddot{s} = -32$. Partial derivatives are often denoted by a **subscript notation** indicating the independent variables. For example, with the subscript notation the second equation in (3) becomes $u_{xx} = u_{tt} - 2u_t$.

**CLASSIFICATION BY ORDER** The **order of a differential equation** (either ODE or PDE) is the order of the highest derivative in the equation. For example,

second order first order

$$\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right)^3 - 4y = e^x$$

is a second-order ordinary differential equation. First-order ordinary differential equations are occasionally written in differential form $M(x, y)\,dx + N(x, y)\,dy = 0$. For example, if we assume that $y$ denotes the dependent variable in $(y - x)\,dx + 4x\,dy = 0$, then $y' = dy/dx$, so by dividing by the differential $dx$, we get the alternative form $4xy' + y = x$. See the *Remarks* at the end of this section.

In symbols we can express an $n$th-order ordinary differential equation in one dependent variable by the general form

$$F(x, y, y', \ldots, y^{(n)}) = 0, \tag{4}$$

where $F$ is a real-valued function of $n + 2$ variables: $x, y, y', \ldots, y^{(n)}$. For both practical and theoretical reasons we shall also make the assumption hereafter that it is possible to solve an ordinary differential equation in the form (4) uniquely for the

*Except for this introductory section, only ordinary differential equations are considered in *A First Course in Differential Equations with Modeling Applications,* Ninth Edition. In that text the word *equation* and the abbreviation DE refer only to ODEs. Partial differential equations or PDEs are considered in the expanded volume *Differential Equations with Boundary-Value Problems,* Seventh Edition.

highest derivative $y^{(n)}$ in terms of the remaining $n + 1$ variables. The differential equation

$$\frac{d^n y}{dx^n} = f(x, y, y', \ldots, y^{(n-1)}), \tag{5}$$

where $f$ is a real-valued continuous function, is referred to as the **normal form** of (4). Thus when it suits our purposes, we shall use the normal forms

$$\frac{dy}{dx} = f(x, y) \qquad \text{and} \qquad \frac{d^2y}{dx^2} = f(x, y, y')$$

to represent general first- and second-order ordinary differential equations. For example, the normal form of the first-order equation $4xy' + y = x$ is $y' = (x - y)/4x$; the normal form of the second-order equation $y'' - y' + 6y = 0$ is $y'' = y' - 6y$. See the *Remarks*.

**CLASSIFICATION BY LINEARITY** An $n$th-order ordinary differential equation (4) is said to be **linear** if $F$ is linear in $y, y', \ldots, y^{(n)}$. This means that an $n$th-order ODE is linear when (4) is $a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y - g(x) = 0$ or

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x). \tag{6}$$

Two important special cases of (6) are linear first-order ($n = 1$) and linear second-order ($n = 2$) DEs:

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \qquad \text{and} \qquad a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x). \tag{7}$$

In the additive combination on the left-hand side of equation (6) we see that the characteristic two properties of a linear ODE are as follows:

- The dependent variable $y$ and all its derivatives $y', y'', \ldots, y^{(n)}$ are of the first degree, that is, the power of each term involving $y$ is 1.
- The coefficients $a_0, a_1, \ldots, a_n$ of $y, y', \ldots, y^{(n)}$ depend at most on the independent variable $x$.

The equations

$$(y - x)dx + 4x\,dy = 0, \qquad y'' - 2y' + y = 0, \qquad \text{and} \qquad \frac{d^3y}{dx^3} + x\frac{dy}{dx} - 5y = e^x$$

are, in turn, linear first-, second-, and third-order ordinary differential equations. We have just demonstrated that the first equation is linear in the variable $y$ by writing it in the alternative form $4xy' + y = x$. A **nonlinear** ordinary differential equation is simply one that is not linear. Nonlinear functions of the dependent variable or its derivatives, such as $\sin y$ or $e^{y'}$, cannot appear in a linear equation. Therefore

$$\underbrace{(1 - y)y'}_{\substack{\text{nonlinear term:}\\ \text{coefficient depends on } y}} + 2y = e^x, \qquad \frac{d^2y}{dx^2} + \underbrace{\sin y}_{\substack{\text{nonlinear term:}\\ \text{nonlinear function of } y}} = 0, \qquad \text{and} \qquad \frac{d^4y}{dx^4} + \underbrace{y^2}_{\substack{\text{nonlinear term:}\\ \text{power not 1}}} = 0$$

are examples of nonlinear first-, second-, and fourth-order ordinary differential equations, respectively.

**SOLUTIONS** As was stated before, one of the goals in this course is to solve, or find solutions of, differential equations. In the next definition we consider the concept of a solution of an ordinary differential equation.

**DEFINITION 1.1.2 Solution of an ODE**

Any function $\phi$, defined on an interval $I$ and possessing at least $n$ derivatives that are continuous on $I$, which when substituted into an $n$th-order ordinary differential equation reduces the equation to an identity, is said to be a **solution** of the equation on the interval.

In other words, a solution of an $n$th-order ordinary differential equation (4) is a function $\phi$ that possesses at least $n$ derivatives and for which

$$F(x, \phi(x), \phi'(x), \ldots, \phi^{(n)}(x)) = 0 \qquad \text{for all } x \text{ in } I.$$

We say that $\phi$ *satisfies* the differential equation on $I$. For our purposes we shall also assume that a solution $\phi$ is a real-valued function. In our introductory discussion we saw that $y = e^{0.1x^2}$ is a solution of $dy/dx = 0.2xy$ on the interval $(-\infty, \infty)$.

Occasionally, it will be convenient to denote a solution by the alternative symbol $y(x)$.

**INTERVAL OF DEFINITION** You cannot think *solution* of an ordinary differential equation without simultaneously thinking *interval*. The interval $I$ in Definition 1.1.2 is variously called the **interval of definition,** the **interval of existence,** the **interval of validity,** or the **domain of the solution** and can be an open interval $(a, b)$, a closed interval $[a, b]$, an infinite interval $(a, \infty)$, and so on.

## EXAMPLE 1 Verification of a Solution

Verify that the indicated function is a solution of the given differential equation on the interval $(-\infty, \infty)$.

**(a)** $dy/dx = xy^{1/2}; \quad y = \frac{1}{16}x^4$ **(b)** $y'' - 2y' + y = 0; \quad y = xe^x$

**SOLUTION** One way of verifying that the given function is a solution is to see, after substituting, whether each side of the equation is the same for every $x$ in the interval.

**(a)** From

$$\textit{left-hand side:} \quad \frac{dy}{dx} = \frac{1}{16}(4 \cdot x^3) = \frac{1}{4}x^3,$$

$$\textit{right-hand side:} \quad xy^{1/2} = x \cdot \left(\frac{1}{16}x^4\right)^{1/2} = x \cdot \left(\frac{1}{4}x^2\right) = \frac{1}{4}x^3,$$

we see that each side of the equation is the same for every real number $x$. Note that $y^{1/2} = \frac{1}{4}x^2$ is, by definition, the nonnegative square root of $\frac{1}{16}x^4$.

**(b)** From the derivatives $y' = xe^x + e^x$ and $y'' = xe^x + 2e^x$ we have, for every real number $x$,

$$\textit{left-hand side:} \quad y'' - 2y' + y = (xe^x + 2e^x) - 2(xe^x + e^x) + xe^x = 0,$$

$$\textit{right-hand side:} \quad 0.$$

■

Note, too, that in Example 1 each differential equation possesses the constant solution $y = 0$, $-\infty < x < \infty$. A solution of a differential equation that is identically zero on an interval $I$ is said to be a **trivial solution.**

**SOLUTION CURVE** The graph of a solution $\phi$ of an ODE is called a **solution curve.** Since $\phi$ is a differentiable function, it is continuous on its interval $I$ of definition. Thus there may be a difference between the graph of the *function* $\phi$ and the